PHY465 Experimental Astroparticle Physics: Cosmic Rays

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Supplementary notes for the Cosmic Ray (CR) part of the PHY465 Experimental Astroparticle Physics course 2020 at the University of Zurich. These notes comprise mainly derivations and calculations not contained on the lecture slides and are not intended to be a complete set of notes used in isolation.

Contents

1 Cosmic Ray Production

1.1 Cosmic Ray Energy Density

The energy density of CRs in the Milky Way is $\varepsilon_{\rm CR} \simeq 1{\rm eVcm}^{-3}$, which is comparable to the energy densities of the Cosmic Microwave Background (CMB) $\varepsilon_{\rm CMB} \simeq 0.3{\rm eVcm}^{-3}$, and of visible light $\varepsilon_{\rm Vis} \simeq 0.3{\rm eVcm}^{-3}$. The average lifetime of a CR in our Galaxy is $\tau_{\rm CR} \simeq 10^7$ years, as inferred from relative isotopic abundances (see section 4.2). What kinetic power is required in order to sustain this CR energy density?

$$P_{\rm CR} = \frac{\varepsilon_{\rm CR} V_{\rm Gal}}{\tau_{\rm CR}} = 3 \times 10^{33} {\rm W}\,, \tag{1}$$

where for the volume of the Galaxy $V_{\rm Gal}$ a diameter d of 30 kpc and a scale height h of 0.3 kpc has been assumed, such that $V_{\rm Gal} = \frac{\pi}{4} d^2 h = 6 \times 10^{60} {\rm m}^3$ (recall 1 pc $= 3.16 \times 10^{16}$m).

What type of astrophysical accelerator could sustain this power? If we consider supernova remnants, typical energy of the explosion is $E \sim 10^{46}$ J, with a kinetic energy release of $\sim 10^{44}$ J. On average, we observe one supernova per 100 years in our Galaxy, however only approximately one third of our Galaxy is visible to us. Assuming that supernovae occur at an average rate in the Galaxy of one every 30 years, or 10^9 s, then the kinetic energy released by supernovae into the Galaxy is $10^{44}/10^9 = 10^{35}$ W. Therefore, just a few percent of the kinetic energy release by supernovae is sufficient to sustain the CR power in the Galaxy.

1.2 LHC equivalent Energy

When comparing energies between collider experiments and fixed target situations - such as CR impeding Earth, it is important to take the reference frame into account. Recall thee Mandelstam variable $s = (E_1 + E_2)^2$ for the total energy. In a collider system with two opposing beams of the same particle and same energy (e.g. LHC p-p), $\sqrt{s} = 2E$. In a fixed target situation, $\sqrt{s} = \sqrt{2m_2 E_1}$ where m_2 is the mass of the target and E_1 the energy of the beam, which is assumed to be much greater than the particle mass. Hence, for 14 TeV centre-of-mass energy at the LHC, this is equivalent to $E = (14{\rm TeV})^2/(2 \times m_p) = 1 \times 10^{17}$ eV CR energy.

1.3 Hillas Criterion

The maximum energy to which a particle can be accelerated depends on the maximum energy to which the particle remains confined to the accelerating region. For acceleration in a magnetic field, this means that the Larmor radius of the particle R_L must be less than L, the size of the region $R_L \leq L$. The Larmor radius $R_L = \frac{mv}{|q|B} = \frac{p}{|q|B} = \frac{E}{|q|cB}$ where we have used $E = pc$ assuming that the energy is dominated by the momentum. For any mass (fully ionised) particle, $q = Ze$ which results in the Hillas Criterion for the maximum energy:

$$E_{\rm max} = ZecBL\,. \tag{2}$$

The size and magnetic field properties of astrophysical sources can be used to simply determine, via the Hillas criterion, whether they are plausible accelerators of CRs.

1.4 Rankine-Hugoniot Equations

The Rankine-Hugoniot equations describe the conservation of mass, momentum and energy at a shock front.

1. **Mass conservation**:From our test volume, we consider the amount of mass that has traverse the shock within a time Δt. The cross-sectional area A of the volume perpendicular to the shock remains constant. The volume is the same size on either side of the shock. The amount of matter is therefore $Av\Delta t\rho$ where v and ρ are the speed and density of the material respectively, which change either side of the shock. Hence:

$$\begin{aligned} Av_1 \Delta t \rho_1 &= Av_2 \Delta t \rho_2 \\ v_1 \rho_1 &= v_2 \rho_2\,, \end{aligned} \tag{3}$$

which is the first of the Rankine-Hugoniot equations.

2. **Momentum conservation**: Here we consider the Euler equation:

$$\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v = F - \nabla P \tag{4}$$

considering the case of a one-dimensional shock in the steady state, there is no external force F, or time-dependent change. Equation (4) therefore simplifies to:

$$\frac{d}{dx}\left(P + \rho v^2\right) , \tag{5}$$

resulting in the second of the Rankine-Hugoniot equations:

$$p_1 + \rho_1 v_1^2 = p_2 + \rho_2 v_2^2 , \tag{6}$$

3. **Energy conservation**: Next we consider the types of energy involved in the system:

   - Flow kinetic energy: $\frac{1}{2}v^2$ per unit mass (momentum)
   - Internal (thermal) energy: ε_m per unit mass (compare this with dU in thermodynamics)
   - PV work-type energy: from the first law of thermodynamics $dQ = -dU + pdV$, with volume per unit mass $v = \rho^{-1}$

   Combining these various components, we define the enthalpy per unit mass, $w = \varepsilon_m + pV$ (analogous to the thermal $U + pV$). Returning to the test volume, there is equal energy content on both sides:

$$\begin{aligned} \rho_1(\frac{1}{2}v_1^2 + w_1)Av_1\Delta t &= \rho_2(\frac{1}{2}v_2^2 + w_2)Av_2\Delta t \\ \frac{1}{2}v_1^2 + w_1 &= \frac{1}{2}v_2^2 + w_2 \end{aligned} \tag{7}$$

   where we have made use of the first R-H equation (3) in obtaining the third.

1.5 Shock Compression Ratio

By combining the R-H equations, it is possible to obtain an expression for the shock compression ratio which, under the types of strong shocks we are considering, can be shown to be a constant. We will make use of the expressions $w = \frac{\gamma}{\gamma-1}pV$ (where γ is the heat capacity ratio) for the internal energy and $V = 1/\rho$ for the specific volume of an ideal gas.

Firstly, we wish to eliminate ρ_2 and P_2, which can be done by rearranging equations (3) and (6) as:

$$\rho_2 = \rho_1\left(\frac{v_1}{v_2}\right) \tag{8}$$

$$P_2 = P_1 + \rho_1 V_1(v_1 - v_2) \tag{9}$$

and by substituting for w and V in (7), we obtain:

$$\begin{aligned} \frac{1}{2}v_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} &= \frac{1}{2}v_2^2 + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} \\ \frac{1}{2}v_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} &= \frac{1}{2}v_2^2 + \frac{\gamma}{\gamma-1}\frac{P_1 + \rho_1 v_1(v_1 - v_2)}{\rho_1(v_1/v_2)} \end{aligned} \tag{10}$$

Rearranging this expression according to powers of v_2, dividing through by v_1^2 and substituting for P_1 by the sound speed $c_1 = \sqrt{\gamma P_1/\rho_1}$ reduces to a quadratic equation in powers of $t = v_2/v_1$:

$$\begin{aligned} \left(\frac{\gamma+1}{\gamma-1}\right)t^2 - \frac{2\gamma}{\gamma-1}\left(\frac{c_1^2}{v_1^2} + \gamma\right)t + 1 + \frac{2}{\gamma-1}\frac{c_1^2}{v_1^2} &= 0 \\ \left(\frac{\gamma+1}{\gamma-1}\right)t^2 - \frac{2\gamma^2}{\gamma-1}t + 1 &= 0 \\ t^2 - \frac{2\gamma^2}{\gamma+1}t + \frac{\gamma-1}{\gamma+1} &= 0 \end{aligned} \tag{11}$$

where we have assumed that in fast shocks, the shock speed $v \gg c$ the sound speed, such that terms in c^2/v^2 are small. Equation (11) has solutions $t = 1$ or $t = \frac{\gamma-1}{\gamma+1}$. Recall that $t = \frac{v_2}{v_1} = \frac{\rho_1}{\rho_2}$ is the shock compression ratio. For a monatomic gas with $\gamma = 5/3$, $t = 4$ such that $v_2 = 4v_1$ and the relative speed of upstream and downstream material is $\frac{3}{4}U$ (taking $U = v_1$ as the velocity of flow into the shock in the rest frame).

1.6 Thermonuclear Explosion Energy

Using dimensional analysis: in the interstellar medium, initially $E_0 \simeq 10^{44}$J of kinetic energy, with dimensions $[E_0] = ml^2t^{-2}$. The resistive opacity of the medium is dependent on the density, with dimensions $[\rho_{\rm ISM}] = ml^{-3}$. Young SN remnants undergo a phase of self-similar adiabatic expansion, when the ratio of these quantities is fixed, and has dimensions: $\left[\frac{E_0}{\rho_{\rm ISM}}\right] = \frac{ml^2t^{-2}}{ml^{-3}} = l^5t^{-2}$. Hence we can obtain the expression

$$l = \left(\frac{E_0}{\rho_{\rm ISM}}\right)^{\frac{1}{5}} t^{\frac{2}{5}} \tag{12}$$

which describes the expansion with time. This expansion phase in supernova remnants is referred to as the 'Taylor-Sedov-von Neumann' phase, and equation (12) can be used to describe the expansion of any generic thermonuclear fireball. G. I. Taylor famously used this equation to infer the energy of a nuclear explosion (E_0) from a series of timed photographs, enabling him to estimate l and know both t and $\rho_{\rm atm}$.

1.7 Pulsar Properties

During a supernova explosion, if the progenitor star is sufficiently massive, then the central core may be unable to support itself against collapse with electron degeneracy pressure and may collapse further, to form a Neutron Star supported by neutron degeneracy pressure. Neutron stars with jets of emission that swing past Earth as the star rotates, forming detectable 'pulses' of emission are known as Pulsars and form reliable astrophysical clocks. The rotation period and magnetic field strength of pulsars can be estimated by the conservation of angular momentum and of magnetic flux during the collapse process.

Let us consider collapse from an initial stellar core, with similar properties to a White Dwarf; initial period $P_i \approx 10^3$ s and initial radius $R_i \approx 8000$ km. The radii of Neutron Stars is known to be typically $R_f \approx 10$ km.

1. Conservation of angular momentum, $L = I\omega$, with moment of inertia $I = \frac{2}{5}MR^2$ and rotation frequency $\omega = 2\pi/P$ such that $L = \frac{4\pi MR^2}{5P}$. Given that $L_i = L_f$, we obtain that the rotation period after collapse is:

$$P_f = \frac{R_f^2}{R_i^2} P_i \approx 1.5\,\text{ms} . \tag{13}$$

2. Conservation of magnetic flux, $\phi \propto BR^2$, which means that $\phi_i = \phi_f$ and:

$$\begin{aligned} \frac{\phi_i}{\phi_f} &= \frac{B_iR_i^2}{B_fR_f^2} \\ \left(\frac{R_i}{R_f}\right)^2 &= \frac{B_f}{B_i} \end{aligned} \tag{14}$$

such that the magnetic field strength increases by a factor of $\sim 10^6$ during the collapse.

3. Minimum rotation period; this can be approximated as the minimum rotation period for which gravity still holds above the centripetal force due to the rotation $m\omega^2R$, centripetal $<$ gravitational.

$$\begin{aligned} \frac{m4\pi^2R}{P^2} &\leq \frac{GMm}{R^2} \\ P^2 &\geq \frac{4\pi^2R^3}{GM} , \end{aligned} \tag{15}$$

which, for typically assumed neutron star properties of $R \approx 10$ km and $M \approx 1.4M_\odot$, yields a minimum period of 0.46 ms.

2 Cosmic Ray Propagation

2.1 Solar Modulation

Solar modulation is the modification of the CR spectrum arriving at Earth with respect to that in the interstellar medium (ISM) by the loss of energy in overcoming the solar wind. Using a force-field approximation with potential parameter $\phi = 550\,\text{MeV}$, the expression for the modification of the CR spectrum is:

$$\Phi_{1\text{AU}}(E) = \frac{E^2 - m^2}{(E+\phi)^2 - m^2} \times \Phi_{\text{ISM}}(E+\phi) \; , \tag{16}$$

which for protons $m = m_p = 940\,\text{MeV}$ implies that at $1\,\text{GeV}$ only $\sim 8\%$ of the ISM flux Φ_{ISM} survives, whilst at $10\,\text{GeV} \sim 90\%$ survives.

2.2 Cosmic Ray Knee Z-dependence

A note on the CR knee - a characteristic change in spectral index (steepening) of the CR spectrum. Propagation and acceleration effects typically relate to the magnetic field B. Therefore, if a spectral change is related to this effects, then it should occur at a characteristic Rigidity, $R_c = pc/Ze \approx E/Ze$ i.e. at the same value $\sim 3PV$ once the Z dependence has been removed. Equivalently, this can be thought of as a change that occurs for protons at $E = R_c$ will occur for He at $2R_c$, O at $8R_c$ etc. If the proton knee occurs at $\sim 3\,\text{PeV}$, then the Fe knee is expected at $26 \times 3\,\text{PeV}$; this is indeed seen in results from KASCADE-grande measurements of the composition decomposed CR spectra, with an Fe knee occurring at $\log_{10}(E/eV) = 16.9$.

2.3 East-West Effect

At low CR energies, the trajectories of incoming particles are affected by the Earth's magnetic field. Considering a particle with charge Ze orbiting in the equatorial plane of the Earth's dipolar magnetic field, the balance between the centripetal and Lorentz forces is:

$$Ze|v \times B| = \frac{mv^2}{r} \tag{17}$$

using the expression for the magnetic field strength $B = \mu_0 M / 4\pi r^3$ with magnetic moment M ($M = 8 \times 10^{22}$ Am for the Earth), the orbital radius is:

$$r = \left(\frac{\mu_0 Z e M}{4\pi m v} \right)^{1/2} \; , \tag{18}$$

where with $R = R_\oplus$ then $p/Z \approx 60\,\text{GeV}$, which is the minimum momentum needed for a proton to be able to reach Earth from the East.

The sign of this flux asymmetry was used by Rossi et al. to show that CRs are positively charged.

2.4 Diffusion Loss Equation

To characterise the CR population, we define $N(E,x,t)\Delta E \Delta x$ as the number of particles with an energy between E and $E + \Delta E$ and a location between x and $x + \Delta x$ at time t. The rate of change of the particle population, $\frac{\partial N}{\partial t}$ is determined by:

1. Local birth rate $Q(E,x,t)$ due to e.g. acceleration in supernovae
2. Spatial diffusion down concentration gradients $\sim D\nabla^2 N$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ reduces to $\frac{\partial^2}{\partial x^2}$ in the one dimensional case and D is a scalar diffusion coefficient.
3. Energy loss $-\frac{\mathrm{d}E}{\mathrm{d}t} = b(E)$, with the function b depending on the physics of the dominant loss mechanism e.g. for Synchrotron loss $b(E) \sim E^2$. If b is positive, the particles lose energy.

The loss rate and the number of particles change in energy space, such that the rate of change in the number of particles due to energy loss is:

$$\begin{aligned}\left[\frac{\partial N}{\partial t}\right] &= b\left(E+\frac{\Delta E}{2}\right)N\left(E+\frac{\Delta E}{2}\right)-b\left(E-\frac{\Delta E}{2}\right)N\left(E-\frac{\Delta E}{2}\right) \\ &= \Delta E\frac{\partial}{\partial E}[b(E)N(E)] \end{aligned} \tag{19}$$

where the fundamental theorem of calculus has been used in expressing the second line. Hence by including the key factors affecting diffusion, we obtain the Diffusion loss equation for the time evolution of the energy spectrum of the particles:

$$\frac{\partial N}{\partial t} = Q + D\nabla^2 N + \frac{\partial}{\partial E}[b(E)N(E)] \tag{20}$$

where the three terms represent a source of particles, spatial diffusion and energy loss respectively.

This equation can be simplified for high energy particles by making two assumptions: steady state such that $\frac{\partial}{\partial t}=0$; and infinite uniform *distribution of sources*, i.e. that the particles have already diffused and there are no spatial gradients. Therefore we can solve equation (20) for the number of particles with an arbitrary energy:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}E}[b(E)N(E)] &= -Q(E) \\ \int_{E_{\mathrm{arb}}}^{\infty}\mathrm{d}[b(E)N(E)] &= -\int_{E_{\mathrm{arb}}}^{\infty}Q(E)\mathrm{d}E \\ N(E_{\mathrm{arb}}) &= \frac{1}{b(E_{\mathrm{arb}})}\int_{E_{\mathrm{arb}}}^{\infty}Q(E)\mathrm{d}E \end{aligned} \tag{21}$$

where in taking the limits we have made use of the fact that there are no particles with infinite energy. The source term can be expected to *have an injection spectrum which* varies with energy as $Q(E)\sim\kappa E^{-p}$, where p is some number greater than 1, such that:

$$\begin{aligned}\int_{E_{\mathrm{arb}}}^{\infty}Q(E)\mathrm{d}E &= \frac{\kappa}{p-1}E_{\mathrm{arb}}^{1-p} \\ N(E) &= \frac{\kappa E^{-(p-1)}}{(p-1)b(E)} \end{aligned} \tag{22}$$

where we have been able to remove the arbitrary energy subscript and the index p is where the physics of acceleration enters the model, with local conditions affecting the dominant processes. In general, *the form of b is the sum of contributing processes*:

$$b(E) = A_1 + A_2E + A_3E^2 \ , \tag{23}$$

where the three terms correspond to ionisation loss (dependent on the particle energy), bremsstrahlung (or adiabatic losses, dependent on particles doing work $p\mathrm{d}V$) and a combination of synchrotron and inverse Compton processes respectively. The exact values of $A_{1,2,3}$ depend on local parameters such as the density and magnetic field strength. Considering equations (22) and (23), if ionisation loss dominates, $N(E)\sim E^{-(p-1)}$, whilst if synchrotron losses dominate $N(E)\sim E^{-(p+1)}$ *and the spectrum is unchanged if adiabatic losses dominate,* $N(E)\sim E^{-p}$. This is actually a useful tool - by measuring the slope of number of particles as a function of energy, we can draw inferences about the local environment and acceleration physics.

2.5 Ginzburg-Syrovatskii Equation

Terms can be added to equation (20) to describe additional processes; spallation gains and losses and radioactive decay. The diffusion loss equation for nuclear species i, also called the Ginzburg-Syrovatski equation is:

$$\frac{\partial N_i}{\partial t} = Q + D\nabla^2 N_i + \frac{\partial}{\partial E}[b(E)N_i] - \frac{N_i}{\tau_i} - \frac{N_i}{\tau_{ri}} + \sum_{j>i}\frac{P_{ji}N_j}{\tau_j} \tag{24}$$

where the last term is another source term with P_{ji} being the probability that in a spallation of j, (an inelastic collision involving the destruction of nucleus j) species i is created, τ_i is the average lifetime against being broken by spallation and τ_{ri} is the radioactive lifetime. This is often considered in the context of the so-called Leaky Box Model, which considers the change in particle population within a test volume.

(See also the discussion of Cosmic Ray clocks in section 4.2.)

2.6 GZK Cut-off

Greisen, Zatsepin and Kuzmin realised soon after the discovery of the CMB, that these photons make the universe opaque to high energy CRs - the GZK effect. The energy threshold for pion production through $\gamma + p \rightarrow \pi^0 + p$ is given by:

$$\begin{aligned} s_{th} &= (m_p + m_{\pi^0})^2 \\ &= m_p^2 + 2E_{th}\varepsilon(1 - \beta_p \cos\theta) \qquad (25) \\ &= m_p^2 + 2\varepsilon'_{th} m_p \qquad (26) \end{aligned}$$

where in the CMB rest frame, equation (25), with photon energy $\varepsilon \sim 10^{-3}$eV, the threshold proton energy for pion production is $E_{th} \sim 7 \times 10^{19}$eV, such that higher energy protons will lose energy via this process until they drop below the threshold energy. In the proton rest frame, equation (26), the photon energy is $\varepsilon'_{th} \approx 145\,\text{MeV}$.

A similar process, Bethe-Heitler pair production of e^+e^- pairs has with $s_{th} = (m_p + 2m_e)^2$ an energy threshold of $E_{th} \sim 6 \times 10^{17}$eV.

3 Extensive Air Showers and Ground-based Detectors

3.1 Heitler Model of shower development

Firstly, we consider leptonic air showers, initiated by energetic γ-rays and electrons. The dominant interactions are pair-production $\gamma + A \rightarrow e^+ + e^- + A$ and bremsstrahlung $e^\pm + A \rightarrow e^\pm + \gamma + A$. The *Radiation Length* is defined as the path length X_0 over which a particles energy reduced by a factor $1/e$. In air, the radiation length is $X_0 = 36.6$g/cm^2 which corresponds to the amount of atmosphere the particle has traversed (note that the density of the atmosphere changes significantly with altitude). The typical path length for pair-production is slightly longer at $\bar{\lambda} = \frac{9}{7}X_0$. Major assumptions used to simplify EAS development in the Heitler model include:

- the radiation lengths for bremsstrahlung and pair-production are the same (in fact not, X_0 vs $\frac{9}{7}X_0$).
- the energy is evenly split between particles (in fact usually not evenly split in pair-production)
- shower development continues until the average energy drops below $E_c = 85\,\text{MeV}$ (for leptonic EAS).

After n radiation lengths (or equivalently travelling a distance x through the atmosphere) the number of particles in the shower is: $N = 2^n = 2^{x/X_0}$ each with energy $E(x) = E_0 2^{-x/X_0}$. The number of particles at shower maximum is $N_{\max} = E_0/E_c = 2^n$ and the depth of shower maximum is $X_{\max} = X_0 + N_{\max} X_{1/2}$ where $X_{1/2} = X_0 \ln 2$.

3.2 Hadronic EAS

For hadronic EAS, the Heitler model considers only the production of pions as the dominant process, with approximately $2/3\pi^\pm$ and $1/3\pi^0$ produced. These decay as $\pi^+ \rightarrow \mu^+ + \nu_\mu$ or $\pi^0 \rightarrow \gamma + \gamma$ on timescales of 10^{-8}s and 10^{-16}s respectively. Given this division of energy, the amount of energy entering the electromagnetic component is $E_{\text{em}} = [1 - (2/3)^n]E_0$ after n radiation lengths. Correspondingly, the energy in the hadronic component is $E_{\text{had}} = [(2/3)^n]E_0$. For hadronic showers, the radiation length is $X_{\text{had}} \approx 120$g/cm^2 with critical energy $E_c = 20\,\text{GeV}$

Note that measurements of the ratio of atmospheric muon species find that $\mu^+/\mu^- \approx 1.2$ as protons, by far the most dominant of CR species, comprise the quark combination uud such that more $\pi^+ = u\bar{d}$ tend to be produced than $\pi^- = d\bar{u}$.

As the mass of the primary particle increases, the depth of shower maximum $X_{\max}$ moves towards higher altitudes, shower fluctuations reduce and more muons are produced. In this model, primary particles with a number of nucleons A are treated as a superposition of A proton showers, such that:

$$N^A_{\text{en,max}}(E_0) = AN^n_{\text{en,max}}(E_n/E_c)X^A_{\max}(E_0) = X_{\max}(E_0/A)N^A_\mu(E_0) = A\left(\frac{E_0/A}{E_{\text{dec}}}\right)^\alpha = A^{1-\alpha}\left(\frac{E_0}{E_{\text{dec}}}\right) \tag{27}$$

4 Extras

4.1 Cosmic Ray Acceleration Power Law

This derivation is far more involved than needed for this course - I am including it only for completeness; feel free to skip.

A shock occurs if a system is perturbed faster than it can coherently respond. In the strong shock regime, $v_1 = \frac{v_2}{4}$ and the relative speed of the shocked and unshocked gas is $U - \frac{U}{4} = \frac{3}{4}U$. In the rest frame of the upstream gas, the downstream gas is seen to be approaching at $\frac{3}{4}U$ and vice versa. (In the rest frame of the downstream gas, the upstream gas approaches at $\frac{3}{4}U$.)

A bulk fluid description has been used, with the average motion over many particles; the bulk flow may be described by a Maxwellian velocity distribution, which is dependent on the mean velocity. Under a kinetic description, individual particles can work against the bulk flow; typically these are velocity outliers, as the particle may cross the shock front. As it does so, it will be scattered, switching from being a member of one Maxwellian distribution to a member of another different one. This causes the particle to lose any memory of its original direction, in a process termed *Isotropisation*, occurring at high energy. The particle may subsequently cross the boundary again, being again isotropised, but without losing energy. Undergoing this process many times may energise the particles.

A particle with initial momentum p_x and energy E may: (i) leave the upstream gas; (ii) cross the shock front and; (iii) enter the downstream gas which is approaching at $\frac{3}{4}U$. As the particle crosses the shock a Lorentz transformation must be applied, such that when the particle has passed into the downstream gas its energy is:

$$E'\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}(E + p_x v)\,, \tag{28}$$

using the Lorentz γ factor. Shock speeds are typically $10-20\times10^3\text{ms}^{-1}$ and non-relativistic with $\gamma \simeq 1$. Assuming the particles is itself relativistic, $E = pc$ the particle momentum in the x-direction becomes $p_x = \frac{E}{c}\cos\theta$ and hence

$$\begin{aligned} E' - E \equiv \Delta E &= \frac{E}{c}v\cos\theta \\ \frac{\Delta E}{E} &= \frac{v}{c}\cos\theta \end{aligned} \tag{29}$$

which gives an expression for the energy change undergone by a single particle as it crosses the shock. The extra energy is initially directed but becomes isotropised by collisions with the downstream gas, so the energy needs to be averaged over the inclination angles of the population; i.e. the probability distribution of ΔE is required. The speed of particles travelling towards the shock along the x-axis is $c\cos\theta$. Combining the solid angle expression, the x speed component and an unknown normalisation constant A, the probability of crossing the shock at angle θ is given by:

$$\begin{aligned} P(\theta)\mathrm{d}\theta &= A\cos\theta\sin\theta\mathrm{d}\theta \\ \int_0^{\frac{\pi}{2}} P(\theta)\mathrm{d}\theta = 1 &= A\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\mathrm{d}\theta \end{aligned} \tag{30}$$

where the total probability is required to be equal to 1. Since the right hand integral is equal to $\frac{1}{2}$, this determines the normalisation constant $A = 2$. Utilising both expressions (29) and (30), the average gain in energy by a particle on <u>one</u> trip from upstream to downstream is:

$$\begin{aligned}\left\{\frac{\Delta E}{E}\right\} &= \int_0^{\frac{\pi}{2}} P(\theta)\frac{\Delta E}{E}(\theta)\mathrm{d}\theta \\ &= 2\frac{v}{c}\int_0^{\frac{\pi}{2}} \cos^2\theta\sin\theta\mathrm{d}\theta \\ &= \frac{2}{3}\frac{v}{c} \end{aligned} \tag{31}$$

where the integral over θ takes the value of $\frac{1}{3}$. If the particle has been isotropised then it may return to cross the shock front again, with the same magnitude of energy change. Therefore, in a single upstream $\rightarrow$ downstream $\rightarrow$ upstream circuit, the average gain in energy after two shock crossings is: $\left\{\frac{\Delta E}{E}\right\} = \frac{4}{3}\frac{v}{c}$; but as $v = \frac{3}{4}U$, per "collision with the shock front" $\left\{\frac{\Delta E}{E}\right\} = \frac{U}{c}$, where on long time-scales a 'collision' is defined as being one circuit.

For a significant gain in energy, many crossings need to occur, and the particle must remain in the vicinity of the shock front. In general, the number of particles crossing an arbitrary surface (such as the shock front) per unit time is $\frac{Nc}{4}$, where N is the particle density and all particles are assumed to have the same velocity, c (which is much greater than the speed of the shock). The downstream flux of particles moving away from the shock front is given by $Nv = \frac{NU}{4}$, therefore the fraction of particles which are lost (*per unit time*), travelling downstream and receiving no further kicks, is $\frac{NU/4}{Nc/4} = \frac{U}{c}$. The probability of a particle remaining in the acceleration zone is therefore $p = 1 - \frac{U}{c} = 1 - \left\{\frac{\Delta E}{E}\right\}$.
After one collision (circuit of the shock front), the average energy of the particle is:

$$E = \beta E_0 = E_0\left(1 + \left\{\frac{\Delta E}{E_0}\right\}\right) = E_0\left(1 + \frac{U}{c}\right)\ , \tag{32}$$

where we set $\beta = (1 + \frac{U}{c})$. The probability of a particle undergoing k successive collisions (energy kicks) is p^k, so after k collisions there are:

$$N = N_0 p^k = N_0\left(1 - \frac{U}{c}\right)^k \tag{33}$$

particles with energy $\beta^k E_0$, where a factor of β is obtained from each collision and N_0 is the initial number of particles. Taking logarithms to eliminate k, as the number of collisions is generally unknown, we have $\ln(\frac{N}{N_0}) = k\ln p$ and $\ln(\frac{E}{E_0}) = k\ln\beta$ such that:

$$\ln\left(\frac{N/N_0}{E/E_0}\right) = \frac{\ln p}{\ln\beta} = \frac{\ln(1-\frac{U}{c})}{\ln(1+\frac{U}{c})} \simeq \frac{-U/c}{U/c} = -1 \tag{34}$$

by expanding the relation $\frac{1-x}{1+x}$ for small values of x. Hence we have the relation that $\frac{N}{N_0} = \left(\frac{E}{E_0}\right)^{-1}$, such that for a larger energy E, there are fewer particles which is good for forming a distribution. This quantity is the number of particles whose energy is *at least* E, hence we can form the differential energy spectrum n, with $n(E)\mathrm{d}E$ representing the number of particles with energy between E and $E + \mathrm{d}E$, leading to:

$$\begin{aligned}\frac{1}{N_0}\int_E^{\infty} n(E)\mathrm{d}E &= \left(\frac{E}{E_0}\right)^{-1} \\ n(E)\mathrm{d}E &= \text{constant} \times \left(\frac{E}{E_0}\right)^{-2}\mathrm{d}E \end{aligned} \tag{35}$$

which is a power law relation. On a log-log plot, (n versus E) this would given a negative slope of gradient 2.

4.2 Cosmic Ray Clocks

An alternative way of utilising equation (24) is to ask how much time must have elapsed to generate the observed elemental abundance. Consider radioactive Beryllium-10, an effective CR clock produced in spallation. $\sim 10\%$ of Be is formed as ${}^{10}_{4}$Be, which is unstable to β decay:

$$ {}^{10}_{4}\mathrm{Be} \rightarrow {}^{10}_{5}\,\mathrm{B} + e^{-} + \overline{\nu} $$

with a half life $\tau_r \sim 3.9 \times 10^6$ years, comparable to the confinement time τ_c inferred from the abundance ratio. From the Leaky Box model with the Ginzburg-Syrovatskii equation, we seek to describe a steady state with spallation (i.e. no terms in $\mathrm{d}/\mathrm{d}t$ and the source term is for the birth of species i from heavier nuclei via spallation).

$$ -\frac{N_i}{\tau_e(i)} - \frac{N_i}{\tau_{\mathrm{spal}}(i)} - \frac{N_i}{\tau_r(i)} + \sum_{j>i} \frac{P_{ji}}{\tau_{\mathrm{spal}}(j)} N_j = 0\ . \tag{36} $$

Our procedure will now be to solve this equation for (a) non-radioactive isotopes of Beryllium, and (b) ${}^{10}_{4}$Be, and then take the ratio of these solutions. These two cases have similar dynamics since they have similar charge to mass ratios.

For both (a) and (b) assume that $\tau_{\mathrm{spal}} \gg \tau_e$, i.e. that escape occurs sooner than destruction by spallation. For (a), with the source term $C_a = \sum\limits_{j>i} \frac{P_{ij}}{\tau_j} N_j$:

$$ -\frac{N_a}{\tau_e(a)} + C_a = 0 \tag{37} $$

such that the number of non-radioactive isotopes is $N_a = C_a \tau_e(a)$, where τ_e is the lifetime against spatial loss, or the *escape time*. For case (b), the corresponding equation reads:

$$ \begin{aligned} -\frac{N_b}{\tau_e(b)} - \frac{N_b}{\tau_r(b)} + C_b &= 0 \\ N_b &= C_b \left\{ \frac{1}{\tau_e(b)} + \frac{1}{\tau_r(b)} \right\}^{-1} \\ \frac{N_a}{N_b} &= \frac{C_a}{C_b} \left\{ \frac{\tau_e(a)}{\tau_e(b)} + \frac{\tau_e(a)}{\tau_r(b)} \right\} \end{aligned} \tag{38} $$

$$ \frac{N({}^{7}\mathrm{Be})}{N({}^{10}\mathrm{Be})} = \left\{ 1 + \frac{\tau_e}{\tau_r({}^{10}\mathrm{Be})} \right\} \frac{C({}^{7}\mathrm{Be})}{C({}^{10}\mathrm{Be})} \tag{39} $$

where it has been assumed that the ratio of the escape times is ≈ 1. The abundance ratio of Beryllium isotopes is a measurable quantity and the source ratio is known from laboratory nuclear physics. Similarly, the radioactive half-life of ^{10}Be is known from laboratory experiments, enabling equation (39) to be solved for the only unknown quantity – the escape time τ_e. Measurements yield a value for the escape time of $\tau_e \simeq 10^7$ years.

5 Recommended Texts

A brief list of some recommended text books and other resources.

- M. S. Longair *High Energy Astrophysics*
- T. K. Gaisser, R. Engel & E. Resconi *Cosmic Rays and Particle Physics*
- ...